

Control Systems : Set 13 : LQR - Solutions

Prob 1 | Consider the double integrator system $G(s) = \frac{1}{s^2}$ given in control canonical form

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

a) Give the optimal LQR controller $u = -Kx$ for the following weighting matrices

$$R = 1 \qquad Q = \begin{bmatrix} \alpha^2 - 2\beta & 0 \\ 0 & \beta^2 \end{bmatrix}$$

Solve the ARE for $P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} \alpha^2 - 2\beta & 0 \\ 0 & \beta^2 \end{bmatrix} = 0$$

We get three equations:

$$0 = -P_2^2 + \beta^2$$

$$0 = P_3 - P_1 P_2$$

$$0 = -P_1^2 + \alpha^2 + 2P_2 - 2\beta$$

which we solve to get

$$P = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha\beta \end{bmatrix}$$

The control law is then

$$K = R^{-1}B^T P = [\alpha \quad \beta]$$

b) Give the characteristic equation for the closed-loop system

$$\det(sI - (A - BK)) = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [\alpha \quad \beta] \right| = \left| \begin{bmatrix} \alpha + s & \beta \\ -1 & s \end{bmatrix} \right| = s^2 + \alpha s + \beta$$

c) Choose an α and β such that the closed-loop system is critically damped, with a natural frequency of 5 rad/s.

From the characteristic equation, we see that $\omega_n^2 = 25 = \beta$

Critical damping calls for $\zeta = 1$, and therefore

$$2\zeta\omega_n = 2 \cdot 5 = \alpha = 10$$

Prob 2 | Consider the system

$$G(s) = \frac{\alpha}{s+1}$$

- a) Give a state-space model in control canonical form

$$\begin{aligned}\dot{x} &= -x + u \\ y &= \alpha x\end{aligned}$$

- b) Design a state-feedback controller so that the closed-loop system has a pole at -2

$$\begin{aligned}\det(s - (A - BK)) &= \det(s - (-1 - 1 \cdot K)) = s + 1 + K = 0 \\ \rightarrow s &= -K - 1\end{aligned}$$

Setting our only pole to -2 , we get $K = 1$

- c) Suppose that α changes to 2α , how does the closed-loop pole change? Is this result surprising? Why?

The pole depends only on A and B , so the pole does not change.

Based on our development of PID controllers, one would expect that changing the gain of the system should change the pole location. Therefore, it might seem surprising that this is not the case here.

We can understand this by looking at the difference between the state-feedback case and the output-feedback case we considered in PID.

For this problem, our state is $x = y/\alpha$, and our control law is $u = Kx = \frac{K}{\alpha}y$. From this we see that if α changes, our control law will scale accordingly. This is because we're not doing feedback on the output directly, but on a scaled version of the output, the state.

- d) Design an estimator for the system so that the estimator pole is 5 times faster than the closed-loop pole.

The characteristic equation for the estimator is

$$\begin{aligned}\det(sI - (A - LC)) &= \det(sI - (A - LC)) = s + 1 + L\alpha = 0 \\ \rightarrow s &= -1 - L\alpha = -5 \cdot 2\end{aligned}$$

Therefore, we take $L = \frac{9}{\alpha}$

- e) Add a reference input to the controller so that the output tracks constant inputs in steady-state

Compute the steady-state solution

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ \alpha & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{N} = KN_x + N_u = 1 \cdot \frac{1}{\alpha} + \frac{1}{\alpha} = \frac{2}{\alpha}$$

The controller is now

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu = \left(-1 - \frac{9}{\alpha}\alpha\right)\hat{x} + \frac{9}{\alpha}y + u = -10\hat{x} + \frac{9}{\alpha}y + u$$

$$u = -K\hat{x} + \bar{N}r = -\hat{x} + \frac{2}{\alpha}r$$

- f) Suppose that the controller was designed for α , but the real system has a value $\tilde{\alpha}$. Does your system track the reference?

Hint: Compute the transfer function from the reference to the output for the closed-loop system, and then use the final value theorem.

All values in the above derivations will remain at the value α , as this was the value used in the controller design. The only value that changes to $\tilde{\alpha}$ is for $y = \tilde{\alpha}x$

Our controller is $u = -\hat{x} + \frac{2}{\alpha}r$.

The closed-loop system is

$$\dot{x} = -x + u = -x - \hat{x} + \frac{2}{\alpha}r$$

$$\dot{\hat{x}} = (A - LC)\hat{x} + LCx + Bu = \left(-1 - \frac{9}{\alpha}\alpha\right)\hat{x} + \frac{9}{\alpha}\tilde{\alpha}x - \hat{x} + \frac{2}{\alpha}r$$

Writing this in standard form we get

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 9\frac{\tilde{\alpha}}{\alpha} & -11 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \frac{2}{\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} \tilde{\alpha} & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

To calculate the DC-gain of the system we can convert to transfer function form, and use the final value theorem

$$T(s) = \frac{Y}{R} = C(sI - A)^{-1}B = \begin{bmatrix} \tilde{\alpha} & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 9\frac{\tilde{\alpha}}{\alpha} & -11 \end{bmatrix} \right)^{-1} \frac{2}{\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{\alpha} & 0 \end{bmatrix} \begin{bmatrix} s+11 & -1 \\ 9\frac{\tilde{\alpha}}{\alpha} & s+1 \end{bmatrix} \frac{1}{(s+1)(s+11) + 9\frac{\tilde{\alpha}}{\alpha}} \frac{2}{\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 2\frac{\tilde{\alpha}}{\alpha} \frac{s+10}{s^2 + 12s + 11 + 9\frac{\tilde{\alpha}}{\alpha}}$$

The final value theorem gives the response to a step input of unit magnitude as

$$\begin{aligned}\lim_{t \rightarrow \infty} y(t) &= \lim_{s \rightarrow 0} sT(s) \frac{1}{s} \\ &= \lim_{s \rightarrow 0} 2 \frac{\tilde{\alpha}}{\alpha} \frac{s+10}{s^2 + 12s + 11 + 9\frac{\tilde{\alpha}}{\alpha}} \\ &= 2 \frac{\tilde{\alpha}}{\alpha} \frac{10}{11 + 9\frac{\tilde{\alpha}}{\alpha}}\end{aligned}$$

which we can see will be one only if $\alpha = \tilde{\alpha}$

- g) Add an integrator to the system and repeat the controller and estimator design processes, setting both closed-loop poles to -2 . Show that it tracks reference inputs even if the value of α used in the design is different from that of the real system.

Note: Use Matlab with various values of α and $\tilde{\alpha}$ to verify the result with the `dcgain` function, or by plotting step responses, etc. If you do it with the final value theorem as in the last question, you'll have to compute a 3×3 matrix inverse. This can also be done with the Matlab symbolic toolbox if you prefer.

We add an integrator to the system by augmenting the state. The resulting augmented model is

$$\begin{aligned}\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \\ y &= \begin{bmatrix} \alpha & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}\end{aligned}$$

We design the controller to have two poles at -2 this time

$$\begin{aligned}\det(s - (A - BK)) &= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} -1 & 0 \\ -\alpha & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [K_1 \ K_2]\right)\right) \\ &= s^2 + (K_1 + 1)s - K_2\alpha = (s + 2)^2 = s^2 + 4s + 4\end{aligned}$$

So we choose $K = [3 \quad -4/\alpha]$ and the controller is $u = -3\hat{x} + \frac{4}{\alpha}x_I$

We only need an estimator for the real state of the system, and so we use the same estimator as before $L = \frac{9}{\alpha}$

With the parameter α changed to $\tilde{\alpha}$ the system equations are

$$\begin{aligned}\dot{x} &= -x + u = -x - 3\hat{x} + \frac{4}{\alpha}x_I \\ \dot{x}_I &= r - y = r - \tilde{\alpha}x \\ \dot{\hat{x}} &= (A - LC)\hat{x} + LCx + Bu = \left(-1 - \frac{9}{\alpha}\tilde{\alpha}\right)\hat{x} + \frac{9}{\alpha}\tilde{\alpha}x - 3\hat{x} + \frac{4}{\alpha}x_I\end{aligned}$$

Writing this in standard form we get

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} -1 & \frac{4}{\alpha} & -3 \\ -\tilde{\alpha} & 0 & 0 \\ 9\frac{\tilde{\alpha}}{\alpha} & \frac{4}{\alpha} & -13 \end{bmatrix} \begin{bmatrix} x \\ x_I \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} r$$

$$y = [\tilde{\alpha} \quad 0 \quad 0] \begin{bmatrix} x \\ x_I \\ \hat{x} \end{bmatrix}$$

Converting to transfer function form (with the Matlab symbolic toolbox) we get

$$T(s) = \frac{4s + 40}{\frac{\alpha}{\tilde{\alpha}}s^3 + 14\frac{\alpha}{\tilde{\alpha}}s^2 + (13\frac{\alpha}{\tilde{\alpha}} + 31)s + 40}$$

The final value theorem shows that the response to a step input is

$$\lim_{s \rightarrow 0} sT(s) \frac{1}{s} = 1$$